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Canonical partition functions for parastatistical systems of any order

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A general formula for the canonical partition function for a parastatistical system of any order is derived. The formula expresses the canonical partition functions for these in terms of sums of Schur functions.The only hitherto known result due to Suranyi [Phys. Rev. Lett. **65**, 2329 (1990)] for parasystems of order two is obtained as a special case. Our results apply not only to parastatistics but to all statistics that can be defined on the basis of the permutation group, including those for which no simple definition in terms of the algebra of creation and annihilation operators is possible. [S1063-651X(96)04107-4]

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Parastatistics [1–3] was introduced by Green [1] as a generalization of Bose and Fermi statistics. This generalization, carried out at the level of the algebra of creation and annihilation operators, involves introducing trilinear relations in place of the bilinear relations that characterize Bose and Fermi systems. The Fock space of a para-Bose system of order p, where p is any positive integer, is characterized by the trilinear relations

$$[a_k,\{a_l,a_m\}]=0;\quad [a_k,\{a_l^\dagger,a_m^\dagger\}]=2\delta_{kl}a_m^\dagger+2\delta_{km}a_l^\dagger;$$
$$[a_k,\{a_l^\dagger,a_m\}]=2\delta_{kl}a_m, \tag{1}$$

and the supplementary conditions

$$a_k a_l^\dagger|0\rangle=p\,\delta_{kl}|0\rangle. \tag{2}$$

Similarly, the trilinear relations

$$[a_k,[a_l,a_m]]=0;\quad [a_k,[a_l^\dagger,a_m^\dagger]]=2\delta_{kl}a_m^\dagger-2\delta_{km}a_l^\dagger;$$
$$[a_k,[a_l^\dagger,a_m]]=2\delta_{kl}a_m, \tag{3}$$

together with the supplementary conditions (2) define para-Fermi systems of order p. Bose and Fermi statistics arise from these as a special case corresponding to $p=1$. A convenient representation of para systems is provided by the Green decomposition. Here the annihilation (creation) operators $a_i(a_i^\dagger)$ for a para system of order p are expressed as sums of annihilation (creation) operators $a_{i\alpha}(a_{i\alpha}^\dagger)$, which carry an extra label α taking values $1,\dots,p$,

$$a_i=\sum_{\alpha=1}^{p}a_{i\alpha},\quad a_i^\dagger=\sum_{\alpha=1}^{p}a_{i\alpha}^\dagger;\quad a_{i\alpha}|0\rangle=0. \tag{4}$$

The operators $a_{i\alpha}$ and $a_{i\alpha}^\dagger$ obey commutation relations that are partly bosonic and partly fermionic. For a para-Bose system of order p these anomalous commutation relations are

$$[a_{i\alpha},a_{j\alpha}]=0;\quad [a_{i\alpha},a_{j\alpha}^\dagger]=\delta_{ij},$$
$$\{a_{i\alpha},a_{j\beta}\}=\{a_{i\alpha},a_{j\beta}^\dagger\}=0\quad \text{if}\ \alpha\neq\beta. \tag{5}$$

For a para-Fermi system of order p, the corresponding relations are

$$\{a_{i\alpha},a_{j\alpha}\}=0;\quad \{a_{i\alpha},a_{j\alpha}^\dagger\}=\delta_{ij},$$
$$[a_{i\alpha},a_{j\beta}];\quad [a_{i\alpha},a_{j\beta}^\dagger]=0\quad \text{if}\ \alpha\neq\beta. \tag{6}$$

The definition of parasystems via the Green decomposition is more amenable to physical interpretations and possible applications than that based on the trilinear relations given above. Indeed, by interpreting α as a new quantum number a model for quarks was proposed by Greenberg [4] as a possible way to overcome certain difficulties with the symmetry properties of three quark wave functions.

Recent developments in interacting many-particle systems have shown that the quasiparticles in such systems may exhibit features far more exotic than those permitted to elementary particles and have led to the advent of fractional statistics [5–7], which interpolate between Bose and Fermi statistics. Of these the anyon statistics [5], based on one-dimensional representations of the braid group, arises in the context of effectively two-dimensional condensed matter systems. The anyon stastistics is peculiar to two-dimensional systems and a possible generalization of the notion of fractional statistics to any dimension has been proposed by the Haldane [6]. In view of the rich variety of statistics that the quasiparticles may exhibit, it appears quite possible that parastatistcs, though originally intended for elementary particles, may be realized in condensed matter physics via the Green decomposition. However, in seeking such applications of parastatistics, it is essential that one has a complete knowledge of the thermodynamic properties of ideal para systems. Often the nature of quasiparticles is deduced by comparing the experimentally observed thermodynamic properties with those of known model systems. It seems surprising that, though parastatistics has been around for over four decades, the first calculation of the canonical partition function for a nontrivial parasystem, a parasystem of order two, was reported only a few years ago [8]. The aim of the present work is to complete this task for a parastatistical system of any order. Our work encompasses not only parastatistics of any order but also all statistics that can be defined on the basis of the permutation group including those for which no simple definition in terms of the algebra of creation and annihilation operators is possible. This is achieved by following the approach to parastatistics pioneered by Messiah and Greenberg [9] and further investigated by Hartle, Stolt, and Taylor [10]. In this approach parastatistics arises in the quantum mechanical description of an assembly of N-identical particles with the permutation group S_N playing a central role in defining various kinds of statistics including the parastatistics of Green.

We begin with the Maxwell-Boltzmann or the infinite statistics and show how various permutation statistics arise from it. Consider a Hilbert space $\mathcal{H}$ built by an N-fold tensor product of a Hilbert space H of dim$M$. Let $1,2,3,\ldots,M$ denote the basis vectors of H and $\epsilon_1,\epsilon_2,\ldots,\epsilon_M$ the associated energies. The M^N basis vectors of $\mathcal{H}$ correspond to each term in the product

$$\underbrace{(1+2+\cdots+M)(1+2+\cdots+M)\cdots(1+2+\cdots+M)}_{N\text{ factors}}. \tag{7}$$

One may consider two decompositions of this set of M^N states.

1. Decomposition based on occupation numbers: This decomposition is required for defining the canonical partition function for an ideal system. Here one groups together states that have the same number of 1's, 2's, etc., regardless of their location in the product. Each such group is characterized by a composition of N, i.e., by a set of occupation numbers $n\equiv(n_1,n_2,\ldots,n_M)$, adding up to N, which give the number of times $1,2,\ldots,M$ occur in the states in that group. Elementary combinatorial considerations tell us that each such group contains $N!/n_1!n_2!\cdots n_M!$ states. The canonical partition function is therefore given by

$$Z_N^{\rm inf}(x_1,\ldots,x_M)=\sum_{\substack{n_i\\ \Sigma n_i=N}}\frac{N!}{n_1!n_2!\cdots n_M!}x_1^{n_1}x_2^{n_2}\cdots x_M^{n_M}, \tag{8}$$

where $x_i\equiv\exp(-\beta\epsilon_i);i=1,\ldots,M$. Using the fact that $N!/n_1!n_2!\cdots n_M!$ is a symmetric function of $n_1,n_2,\ldots,n_M$ we may rewrite the sum over compositions of N in (8) in terms of a sum over partitions of N

$$Z_N^{\rm inf}(x_1,\ldots,x_M)=\sum_{\substack{\lambda\\ |\lambda|=N}}\frac{N!}{\lambda_1!,\ldots,\lambda_M!}m_\lambda(x_1,\ldots,x_M). \tag{9}$$

Here $\lambda\equiv(\lambda_1,\lambda_2,\ldots,\lambda_M)$, $\lambda_1\geqslant\lambda_2\geqslant\lambda_3\cdots\geqslant\lambda_M$ is a partition of N (indicated by $|\lambda|=N$), and $m_\lambda(x_1,\ldots,x_M)$ denotes the monomial symmetric function [11] corresponding to the partition λ:

$$m_\lambda(x_1,\ldots,x_M)=\sum x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_M^{\lambda_M}. \tag{10}$$

The sum on the right-hand side (rhs) of (10) is over all distinct permutations of $(\lambda_1,\ldots,\lambda_M)$.

The sum in (8) can be carried out using the multinomial theorem and the result is

$$\begin{aligned}Z_N^{\rm inf}(x_1,\ldots,x_M)&=(x_1+\cdots+x_M)^N\\ &=\sum_{\substack{\lambda\\ |\lambda|=N}}\frac{N!}{\lambda_1!,\ldots,\lambda_M!}m_\lambda(x_1,\ldots,x_M).\end{aligned} \tag{11}$$

Stated in words, (11) tells us that the contribution of each partition λ to the partition function is given by $m_\lambda(x_1,\ldots,x_M)$ times the number of states $N!/\lambda_1!,\ldots,\lambda_M!$ in that partition. It may also be noted here that the monomial symmetric functions play a special role in the context of decompositions based on occupation numbers. Given the canonical partition function, its expansion in terms of the monomial symmetric functions yields all information regarding the decomposition based on occupation numbers. For instance, setting $x_1=x_2=x_M=1$ in (11) we obtain

$$M^N=\sum_\lambda\frac{N!}{\lambda_1\ldots\lambda_M!}m_\lambda(1,\ldots,1), \tag{12}$$

which tells us that each partition $(\lambda_1,\ldots,\lambda_M)$ corresponds to $m_\lambda(1,\ldots,1)$ sets of occupation numbers obtained by distinct permutations of λ_i's and each such set contains $N!/\lambda_1!\ldots\lambda_M!$ states. For a given λ the number $m_\lambda(1,\cdots,1)$ is given by $M!/m_1!m_2!\cdots$, where m_i denotes the number of times λ_i occurs in the partition λ.

2. Decomposition based on the permutation group: In this decomposition we regard the M^N states as the carrier space for an M^N dimensional representation of the permutation group S_N. This reducible representation can be decomposed into the irreducible representations of S_N which, as is well known, are in one-to-one correspondence with the partitions of N. All features of this decomposition can be deduced from the partition function $Z_N^{\rm inf}(x_1,\ldots,x_M)$ as follows. Using the Frobenius formula, (11) may be written as

$$\begin{aligned}Z_N^{\rm inf}(x_1,\ldots,x_M)&=(x_1+\cdots+x_M)^N\\ &=\sum_{\substack{\lambda\\ |\lambda|=N}}n(\lambda)s_\lambda(x_1,\ldots,x_M),\end{aligned} \tag{13}$$

where $n(\lambda)$ denotes the dimension of the irreducible representation λ of S_N and $s_\lambda(x_1,\ldots,x_M)$ denote the Schur functions [11,12].

$$s_\lambda(x_1,\ldots,x_M)=\frac{\det(x_i^{\lambda_j+M-j})}{\det(x_i^{M-j})};\quad 1\leqslant i,j\leqslant M. \tag{14}$$

Note that the Schur functions, like the monomial symmetric functions, are symmetric functions and can be defined in many different ways. The definition given above is the one that Schur originally used. We shall encounter other definitions of these functions later. The relation (13) will play an important role later and it may be interpreted as follows. The contribution of each irreducible representation λ of S_N to the partition function is equal to the Schur function $s_\lambda(x_1,\ldots,x_M)$ times the number of states in the irreducible representation λ, i.e., its dimension $n(\lambda)$. Thus we see that in this decomposition the Schur functions play the same role as the monomial symmetric functions play in the decomposition based on the occupation numbers. For instance, setting $x_1 = x_2 = \cdots = x_M = 1$ in (13) we get

$$M^N = \sum_{\substack{\lambda \\ |\lambda|=N}} n(\lambda) s_\lambda(1,\ldots,1), \tag{15}$$

which tells us that $s_\lambda(1,\ldots,1)$ is the number of times the irreducible representation λ occurs in this decomposition. The number $s_\lambda(1,\ldots,1)$ is given by

$$s_\lambda(1,\ldots,1) = \prod_{i<j}^{M} \frac{(\lambda_i - \lambda_j + j - i)}{(j-i)}. \tag{16}$$

So far we have been dealing with $\mathcal{H}$ in which all the M^N states were considered as independent. Following Refs. [9] and [10] we now construct out of it a generalized ray space $\mathcal{H}_{\text{phy}}$ by (a) admitting only those operators on $\mathcal{H}$ that are permutation symmetric and (b) identifying those states in $\mathcal{H}$ that have the same expectation values for all permutation symmetric operators.

These assumptions, by Schur's lemma, imply that all states in $\mathcal{H}$ belonging to an irreducible representation λ of S_N count as one state of $\mathcal{H}_{\text{phy}}$. This together with the interpretation of (13) stated above implies that in writing down the partition function appropriate to $\mathcal{H}_{\text{phy}}$ we should take the contribution of each irreducible representation λ of S_N not as $n(\lambda)s_\lambda(x_1,\ldots,x_M)$ but as $s_\lambda(x_1,\ldots,x_M)$. The partition function appropriate to $\mathcal{H}_{\text{phy}}$ is thus given by

$$Z_N^{\text{HST}}(x_1,\ldots,x_M) = \sum_{\substack{\lambda \\ |\lambda|=N}} s_\lambda(x_1,\ldots,x_M). \tag{17}$$

This is the key result of this work. (Here we use the superscripts HST to denote Hartle, Stolt, and Taylor in honor of their contributions to parastatistics.)

We may arrive at the above result from the decomposition (11) as well. In this decomposition each λ corresponds to $N!/\lambda_1!\cdots\lambda_M!$ states, which provide a reducible representation of S_N of dimension $N!/\lambda_1!\cdots\lambda_M!$. Decomposing it into the irreducible representations of S_N and using a known mathematical result [12] we obtain

$$\frac{N!}{\lambda_1!,\ldots,\lambda_M!} = \sum_\chi K_{\chi\lambda} n(\chi), \tag{18}$$

where $K_{\chi\lambda}$ denote the Kostka numbers [11,12]. Using (18) in (11) we can rewrite (11) as

$$Z_N^{\text{inf}}(x_1,\ldots,x_M) = \sum_\chi n(\chi) \sum_\lambda K_{\chi\lambda} m_\lambda(x_1,\ldots,x_M). \tag{19}$$

Following the same logic as above, and setting $n(\chi)=1$ we obtain the following expression for Z_N^{HST} in terms of the monomial symmetric functions:

$$Z_N^{\text{HST}}(x_1,\ldots,x_M) = \sum_\lambda \left(\sum_\chi K_{\chi\lambda} \right) m_\lambda(x_1,\ldots,x_M), \tag{20}$$

which, in view of an alternative definition of Schur functions [11,12] given by

$$s_\chi(x_1,\ldots,x_M) = \left(\sum_\lambda K_{\chi\lambda} \right) m_\lambda(x_1,\ldots,x_M) \tag{21}$$

is easily seen to be the same as (17). The above expression for Z_N^{HST} in terms of the monomial symmetric functions gives us a complete picture of the occupation number decomposition of $\mathcal{H}_{\text{phy}}$. The number of states corresponding to a set of occupation numbers $(\lambda_1,\ldots,\lambda_M)$ (or any distinct permutation thereof) is given by $(\Sigma_\chi K_{\chi\lambda})$. As to the Kostka numbers $K_{\lambda\chi}$ that appear in the above equations, there is a simple combinatorial algoritham to compute them. For given partitions $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_M)$ and $\chi=(\chi_1,\chi_2,\ldots,\chi_M)$ of N the Kostka number $K_{\lambda\chi}$ is equal to the number of ways in which the Young tableau corresponding to the partition λ can be filled up with χ_1 1's, χ_2 2's, etc. in such a way that the numbers along the rows when read from left to right do not decrease and the numbers along the columns when read from top to bottom increase. Thus, for instance, for the partitions $\lambda=(4,2)$ and $\chi=(3,2,1)$ of 6 we get $K_{\lambda\chi}=2$.

So far no restrictions have been put on λ — the sum on the rhs of (17) is over all partitions of N. We shall refer to this statistics as HST statistics. The para-Bose case of order p arises when we restrict the sum in (17) to only those partitions of N whose length $l(\lambda)$ (the number of the nonzero λ_i's) is less than or equal to p. In terms of Young tableaux, this amounts to retaining only those irreducible representations of S_N in which the number of boxes in the first column is $\leqslant p$. The partition function for this case is

$$Z_N^{\text{PB}}(x_1,\ldots,x_M;p) = \sum_{\substack{\lambda \\ |\lambda|=N \\ l(\lambda)\leqslant p}} s_\lambda(x_1,\ldots,x_M). \tag{22}$$

Similarly, the para-Fermi case of order p arises when we restrict λ in (17) to those partitions for which $\lambda_1 \leqslant p$, or, in the language of partitions, to those partitions whose conjugate partition λ' is of length $\leqslant p$. In terms of Young tableaux this implies retaining only those irreducible representations of S_N in which the number of boxes in the first row is $\leqslant p$. The partition function appropriate to this case is

$$Z_N^{\text{PF}}(x_1,\ldots,x_M;p) = \sum_{\substack{\lambda \\ |\lambda|=N \\ l(\lambda')\leqslant p}} s_\lambda(x_1,\ldots,x_M). \tag{23}$$

Likewise, for the (p,q) statistics the corresponding symmetric function $Z_N^{(p,q)}(x_1,\ldots,x_M)$ is obtained by restricting the sum in (17) to those partitions for which $l(\lambda)\leqslant p$ and $l(\lambda')\leqslant q$. The partition functions in all these cases can be expressed in terms of the monomial symmetric functions as

$$Z_N(x_1,\ldots,x_M) = \sum_\lambda \left(\sum_\chi K_{\chi\lambda} \right) m_\lambda(x_1,\ldots,x_M), \tag{24}$$

with χ appropriately restricted.

It may be noted that if $p,q \geqslant N$, all these cases reduce to HST. In other words, the HST statistics is the $p \to \infty$ limit of para-Bose and para-Fermi statistics of order p.

As an illustration, let us consider five para particles of order $p=3$. Using (24) and the table for Kostka numbers given in Ref. [11], we find that the canonical partition function for the para-Bose case is

$$Z_5^{\mathrm{PB}} = m_{(5)} + 2m_{(41)} + 3m_{(32)} + 5m_{(31^2)} + 7m_{(2^2 1)} + 12m_{(21^3)} + 21m_{(1^5)}, \tag{25}$$

and

$$Z_5^{\mathrm{PF}} = m_{(32)} + 2m_{(31^2)} + 4m_{(2^2 1)} + 9m_{(21^3)} + 16m_{(1^5)} \tag{26}$$

for the para-Fermi case. [Here, for brevity, we have omitted the arguments $(x_1, \ldots, x_M)$ and have used a compressed but obvious notation for the partitions.]

The expressions for the partition functions given above are in terms of the Schur and the monomial symmetric functions. One can express them in terms of other symmetric functions as well using some formulas that involve what are known as Jacobi-Trudi determinants. They prove to be extremely useful in carrying out the sums in (22) and (23) with restrictions on the lengths of the partitions and are given by [11,12],

$$s_\lambda(x_1, \ldots, x_M) = \det(h_{\lambda_i - i + j}), \quad 1 \leqslant i,j \leqslant l(\lambda), \tag{27}$$

$$s_\lambda(x_1, \ldots, x_M) = \det(e_{\lambda_i' - i + j}), \quad 1 \leqslant i,j \leqslant l(\lambda'). \tag{28}$$

Here the complete symmetric functions $h_r(x_1, \ldots, x_M)$ and the elementary symmetric functions $e_r(x_1, \ldots, x_M)$ are defined as follows:

$$h_r(x_1, \ldots, x_M) = \sum_{\substack{\lambda \\ |\lambda| = r}} m_\lambda(x_1, \ldots, x_M), \tag{29}$$

$$e_r(x_1, \ldots, x_M) = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}. \tag{30}$$

Using these formulas one can express the partition functions above in terms of either h's or e's, which, as we shall see, are simply the canonical partition functions for bosons and fermions. As an illustration, let us consider the Bose case. Here, since $l(\lambda) \leqslant 1$, we have only one term on the rhs of (22) corresponding to $\lambda = (N,0,0, \ldots, 0)$. Using (24) we obtain

$$Z_N^B(x_1, \ldots, x_M) = h_N(x_1, \ldots, x_M). \tag{31}$$

Similarly, for the Fermi case, one has

$$Z_N^F(x_1, \ldots, x_M) = e_N(x_1, \ldots, x_M). \tag{32}$$

Consider para-Bose of order 2. Using (22) and (24) we obtain

$$Z_N^{\mathrm{PB}}(x_1, \ldots, x_M; 2) = h_N + \sum_{\substack{\lambda_1 + \lambda_2 = N \\ \lambda_1 \geqslant \lambda_2}} \det \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1 + 1} \\ h_{\lambda_2 - 1} & h_{\lambda_2} \end{pmatrix}, \tag{33}$$

which on simplification leads to

$$Z_N^{\mathrm{PB}}(x_1, \ldots, x_M; 2) = \begin{cases} h_P^2(x_1, \ldots, x_M) & \text{if } N = 2P \\ h_{P+1}(x_1, \ldots, x_M) h_P(x_1, \ldots, x_M) & \text{if } N = 2P+1. \end{cases} \tag{34}$$

The result for para-Fermi of order 2 is obtained by replacing h's by e's. Thus we obtain the results due to Suranyi [8], which arise as a special case of (22) and (23).

Finally, for the HST statistics, which, as noted above, is the $p \to \infty$ limit of para-Bose and para-Fermi statistics, the grand canonical partition function can be calculated exactly using a known result [11] for the Schur functions and is given by

$$\mathcal{Z}^{\mathrm{HST}}(x_1, \ldots, x_M) = \prod_i \frac{1}{(1 - x_i)} \prod_{i<j} \frac{1}{(1 - x_i x_j)}, \tag{35}$$

where $x_i = \exp[-(\beta\epsilon_i + \mu)]$. This grand canonical partition function has an interesting structure. It is the product of the grand canonical partition functions of two bosonic systems, one with single-particle energies ϵ_i and the other with single-particle energies $\epsilon_i + \epsilon_j, i \leqslant j$.

To conclude, by adopting the approach propounded in Refs. [9] and [10], and essentially using just the Frobenius formula (13) we have been able to obtain canonical partition functions for all statistics based on the permutation group including those for which no simple second quantized notation is available. In all these statistics the Schur functions play a unifying role. The canonical partitions for all these systems can be expressed as sums of Schur functions with coefficient one. While we have also been able to find an exact expression for the grand canonical partition function for a para-Fermi system of any order, the work on para-Bose systems is still in progress and detailed analyses of the thermodynamic properties derivable from these results would be presented elsewhere.

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